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1994 J. Phys. A: Math. Gen. 27 L797

(http://iopscience.iop.org/0305-4470/27/21/002)

J. Phys. A: Math. Gen. 27 (1994) L797–L800. Printed in the UK

LETTER TO THE EDITOR

Characteristic scales for a scalar field in turbulent flows

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Received 11 April 1994

Abstract. A covariance analysis of a scalar field in 3D and 2D turbulent flows is performed on the basis of the convective–diffusion equation. Physically distinguished scales for coherent structures in the inertial-convective range are obtained.

The dynamics and statistics of turbulent flows are better understood in terms of local characteristics [1], which have internal mechanisms of amplification. For 3D turbulence, the local characteristic of motion is the vorticity field and self-amplification is due to the stretching of vortex filaments [2, 3]. For 2D turbulence, the local characteristic is the vorticity gradient [4–7]. A covariance analysis of 3D vorticity, based on the Navier–Stokes equations, leads to a scale which was associated with 'vortex strings' [8]: $l_s = L\,Re^{-3/10}$, where L is an external scale and Re is the Reynolds number. Similar analysis for 2D vorticity gradients gives a scale for coherent vortex structures [9]: $l_c = L\,Re^{-1/4}$.

In this letter, we extend the analysis of [8, 9] to the case of a passive scalar field $\theta(t, \boldsymbol{x})$ in 3D and 2D turbulent flows. The convection–diffusion equation for incompressible fluid has the form

$$\frac{\partial\theta}{\partial t} + v_k\theta_k = \chi\Delta\theta + q \qquad \theta_k \equiv \frac{\partial\theta}{\partial x_k}\,\frac{\partial v_k}{\partial x_k} = 0. \tag{1}$$

Here $v_k(t, \boldsymbol{x})$ is the velocity field, summation over the repeated indices is from 1 to the dimension of the flow $s = 2, 3$. χ is the diffusivity and q is a statistically stationary source. From (1), we get statistical balance of θ-fluctuations for homogeneous turbulence

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\theta^2\rangle = -N + \langle q\theta\rangle \qquad N = \chi\langle\theta_k^2\rangle. \tag{2}$$

Here $\langle\ \rangle$ means statistical averaging, all fields are taken at the same spacetime location and N is the mean diffusion rate of θ-fluctuations. For a statistically stationary state, the left-hand side (LHS) of balance (2) is zero.

Similarly, spatial differentiation of (1) gives a statistical balance of θ_i

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\theta_i^2\rangle + \langle v_{k,i}\theta_k\theta_i\rangle = -\chi\langle\theta_{ik}^2\rangle + \langle q_i\theta_i\rangle \qquad v_{k,i} \equiv \frac{\partial v_k}{\partial x_i} \qquad \theta_{ik} \equiv \frac{\partial^2\theta}{\partial x_i\,\partial x_k} \qquad q_i \equiv \frac{\partial q}{\partial x_i}. \tag{3}$$

The second term in the LHS of (3) represents the effect of convective amplication of θ_i, namely compression of the fluid element in the direction of θ_i.

0305-4470/94/210797+04$19.50

In order to evaluate the contribution of external source q in balance (3) and for the following analysis, we assume that q is Gaussian and δ-correlated in time. This gives us the formula [8,10] (see also [11] and appendix H in [12])

$$\langle q(\boldsymbol{x})R[\theta(\cdot)]\rangle = \frac{1}{2}\int Q(\boldsymbol{x}'-\boldsymbol{x})\left\langle\frac{\delta R[\theta(\cdot)]}{\delta\theta(\boldsymbol{x}')}\right\rangle \mathrm{d}^s x'. \tag{4}$$

Here, R is any functional of θ, Q is the spatial part of the covariance of q, δ corresponds to the functional derivative and all fields are taken at the same time. From (4)

$$\langle q\theta'\rangle = \tfrac{1}{2}Q(\boldsymbol{r}) \qquad Q(0) = 2N. \tag{5}$$

Here and below, a prime indicates the field taken at the point $\boldsymbol{x}' = \boldsymbol{x}+\boldsymbol{r}$ and we obtain the second equality (5) from balance (2) for the stationary state. From (5), we have

$$\langle q_i\theta_i\rangle = -\tfrac{1}{2}\Delta Q(r)|_{r=0} = sL^{-2}N \tag{6}$$

$$L^{-2} = -\frac{\mathrm{d}^2Q(r)}{\mathrm{d}r^2}\bigg|_{r=0}[Q(0)]^{-1}. \tag{7}$$

Here, we have assumed isotropy and L is the natural external scale, defined by the source [10]. In what follows, we assume, for simplicity, that L is of the same order as the scale at which energy is supplied in 3D turbulence (or enstrophy in 2D turbulence).

Contributions of external source (6) relative to the diffusion term in (3) gives the small parameter

$$\sigma = \left(\frac{l_\mathrm{d}}{L}\right)^2 \qquad l_\mathrm{d}^2 = \frac{\langle\theta_i^2\rangle}{\langle\theta_{ik}^2\rangle} \tag{8}$$

where l_d is the characteristic internal scale of turbulent diffusion. For 3D turbulence with Prandtl number $Pr = \nu/\chi$ of order less than one, we have [11, 12] $l_\mathrm{d} \sim l_\nu = \nu^{3/4}\varepsilon^{-1/4}$, where l_ν is the Kolmogorov internal scale, ν is the kinematic viscosity and ε is the mean rate of energy dissipation. Formula (8) gives

$$\sigma \sim Re^{-3/2} \qquad Re = L^{4/3}\varepsilon^{1/3}\nu^{-1} \tag{9}$$

where Re is the Reynolds number. For $Pr > 1$, parameter σ is even smaller than the expression in (9). Thus, for large Re, the contribution of the external source in balance (3) is negligible and for the stationary state, diffusion is balanced with convective amplification. The same is true for 2D turbulence in the regime of enstrophy cascade [13–15]. In this case, for $Pr \sim 1$, we have

$$l_\mathrm{d} \sim L\,Re^{-1/2} \qquad \sigma \sim Re^{-1} \qquad Re = L^2\gamma^{1/3}\nu^{-1} \tag{10}$$

where γ is the mean rate of enstrophy dissipation—the main parameter in this regime. Similar consideration, with the use of (4), shows that the one-point balance of high-order moments of θ_i is also unaffected by the external source when $Re \gg 1$ (cf vorticity analysis in [8]).

Now we turn to the two-point covariance analysis of the θ-field. Standard procedure [8, 11, 12] gives, for homogeneous isotropic turbulent flow,

$$\frac{\partial}{\partial t}\langle\theta\theta'\rangle + \frac{\partial}{\partial r_k}\langle(v'_k - v_k)\theta\theta'\rangle = 2\chi\Delta\langle\theta\theta'\rangle + Q(r) \tag{11}$$

where we have used (5). By spatial differentiation of (11), we get

$$\frac{\partial}{\partial t}\langle\theta_i\theta'_i\rangle - \Delta\frac{\partial}{\partial r_k}\langle(v'_k - v_k)\theta\theta'\rangle = 2\chi\Delta\langle\theta_i\theta'_i\rangle - \Delta Q(r). \tag{12}$$

For a stationary state, the first term in the LHS of (12) is zero. The second term represents the effect of convective amplification of the θ_i-covariance. The terms in the RHS of (12) correspond to the effects of molecular diffusion and external source.

Assuming that $Q(r)$ is characterized by only one external scale L, we get from (6), for $r \ll L$,

$$\Delta Q(r) \approx -2sL^{-2}N. \tag{13}$$

Consider an inertial-convective range of scales for 3D turbulence

$$\max\{l_\nu, l_\chi\} \ll r \ll L \qquad (l_\chi = \chi^{3/4}\varepsilon^{-1/4}). \tag{14}$$

In this range, neglecting the small intermittency correction, we have [11, 12]

$$\langle\theta\theta'\rangle = c_\theta N\varepsilon^{-1/3}r^{2/3} \tag{15}$$

where c_θ is constant of order one. Thus, having in mind definition (1) of θ_i, we get

$$\Delta\langle\theta_i\theta'_i\rangle = -\Delta^2\langle\theta\theta'\rangle = -\frac{40}{81}c_\theta N\varepsilon^{-1/3}r^{-10/3}. \tag{16}$$

Comparison of the two terms in the RHS of balance (12) shows that when we approach the lower boundary of range (14), the effect of the source becomes negligible for large *Re*. This corresponds to the one-point balance, considered above. However, these two terms (effects of diffusion and external source) become comparable and compensate each other at the scale

$$l_\theta = c(Pr\,Re)^{-3/10} \qquad c = \left(\frac{40c_\theta}{243}\right)^{3/10} \tag{17}$$

which, for most of the practical interesting cases, is within the inertial-convective range (14).

The mere existence of such an intermediate scale suggests that the classical theories of cascade processes in turbulence (and in other dissipative systems with strong interaction) have to be revised. The revision may incorporate coherent structures into the statistical description of turbulence. In particular, the dynamics of boundaries for coherent θ-blobs is determined by the above-indicated effect of convective sharpening of the θ-gradients (θ_i), balanced by the diffusion. Starting from scale (17), the external sources directly affect the dynamics of θ_i-correlations and, thus, affect the coherent structures. Preceding analysis shows that, for scales larger than l_θ, the diffusion is not important and the effect of convection reverses its sign—it reduces θ_i-correlations. It seems that l_θ is a natural physically distinguished scale for coherent θ-structures. For more subtle coherent structures, connected with higher derivatives of the θ-field, the effect of diffusion is stronger and the influence of external sources is shifted to larger scales (see below).

For 2D turbulence in the regime of enstrophy cascade, similar consideration gives (using dimensional argument and neglecting possible logarithmic correction)

$$\Delta\langle\theta_i\theta_i'\rangle \sim N\gamma^{-1/3}r^{-4} \qquad l_\theta^{(2)} \sim L(Pr\,Re)^{-1/4}. \tag{18}$$

Here Re is defined as in (10) and the superscript indicates dimension.

Similarly, the two-point balance for $(m+1)$-order spatial derivatives of the θ-field, obtained by differentiation of (12), gives a hierarchy of increasing scales for 3D turbulence

$$l_\theta(m) \sim L(Pr\,Re)^{-3/(10+6m)} \qquad l_\theta(0) = l_\theta \qquad m = 0, 1, 2, \ldots. \tag{19}$$

The corresponding hierarchy for 2D turbulence is

$$l_\theta^{(2)}(m) \sim L(Pr\,Re)^{-1/(2(2+m))} \qquad l_\theta^{(2)}(0) = l_\theta^{(2)} \qquad m = 0, 1, 2, \ldots. \tag{20}$$

These scales differ from the scales for high-order spatial derivatives of vorticity fields [9] by substitution of $Pr\,Re$ instead of Re. Interpretation of these scales in terms of statistical description of the net of coherent structures can be a subject of a separate study.

We hope that the presented results will stimulate more detailed experimental and numerical studies of coherent structures in turbulent flows. The necessary numerical simulations of turbulence with sufficiently high Re and substantial inertial range will soon be accessible, especially for 2D turbulence. The scalar field is relatively easy to measure (compared to, say, vorticity) by non-intrusive optical methods.

This work is supported by the US Department of Energy under Grant No DE-FG03-91ER14188 with Dr Oscar P Manley as programme manager and by the Office of Naval Research under Grants No ONR-N00014-92-J-1610 and ONR-14-94-1-0040 with Dr Edwin P Rood as programme manager.

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